

On the complexity of the vector connectivity problem

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Abstract

We study a relaxation of the VECTOR DOMINATION problem called VECTOR CONNECTIVITY (VECCON). Given a graph G with a requirement $r(v)$ for each vertex v , VECCON asks for a minimum cardinality set of vertices S such that every vertex $v \in V \setminus S$ is connected to S via $r(v)$ disjoint paths. In the paper introducing the problem, Boros et al. [Networks, 2014, to appear] gave polynomial-time solutions for VECCON in trees, cographs, and split graphs, and showed that the problem can be approximated in polynomial time on n -vertex graphs to within a factor of $\log n + 2$, leaving open the question of whether the problem is NP-hard on general graphs. We show that VECCON is APX-hard in general graphs, and NP-hard in planar bipartite graphs and in planar line graphs. We also generalize the polynomial result for trees by solving the problem for block graphs.

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1. Introduction and background

Recently, Boros et al. [4] introduced the VECTOR CONNECTIVITY problem (VECCON) in graphs. This problem takes as input a graph G and an integer $r(v) \in \{0, 1, \dots, d(v)\}$ for every vertex v of G , and the objective is to find a vertex subset S of minimum cardinality such that every vertex v either belongs to S , or is connected to at least $r(v)$ vertices of S by disjoint paths. If we require each path to be of length exactly 1, we get the well-known VECTOR DOMINATION problem [10], which is a generalization of the famous DOMINATING SET and VERTEX COVER problems.

The VECTOR CONNECTIVITY problem is one of the several problems related to domination and connectivity, which have seen renewed attention in the last few years in connection with the flourishing area of information spreading (see, e.g., [10, 11, 7, 8, 1, 5, 6] and references therein quoted). In a viral marketing campaign one of the problems is to identify a set of targets in a (social) network that can be influenced (e.g., on the goodness of a product) and such that from them most/all the network can be influenced (e.g., convinced to buy the product). The model is based on the assumption that each vertex has a threshold $r(v)$ such that when $r(v)$ neighbors are influenced, also v will get convinced too. Assume now that for v it is not enough that $r(v)$ neighbors are convinced about the product. Vertex v also requires that their motivations are independent. A way to model this “skeptical” variant of influence spreading is to require that each vertex in the network must be reached by $r(v)$ vertex-disjoint paths originating in the target set.

Another scenario where the vector connectivity problem arises is the following: Each vertex in a network produces a certain amount of a given good. We want to place in the network warehouses where the good can be stored. For security/resilience reasons it is better if from each source to each destination (warehouse)

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only a small amount of the good (e.g., one unit) travels at once. In particular, it is preferred if the units of good from one location to the different warehouses travel on different routes. This reduces the risk that if delivery gets intercepted or attacked or interrupted by a fault on the network a large amount of the good gets lost. It is not hard to see that finding the minimum number of warehouses given the amount of units produced at each vertex coincides with the vector connectivity problem.

Boros et al. developed polynomial-time algorithms for VEC CON on split graphs, cographs, and trees, and showed how to model the problem as a minimum submodular cover problem, leading to a polynomial-time algorithm approximating VEC CON within a factor of $\ln n + 2$ on all n -vertex graphs. One of the questions left open in that paper was whether on general graphs, VEC CON is polynomially solvable or NP-hard. In this paper, we answer this question by showing that VEC CON is APX-hard (and consequently NP-hard) in general graphs. Our reduction is from the VERTEX COVER problem in cubic graphs. Simple modifications of the hardness proof allow us to also show that VEC CON is hard in several graph classes for which the VERTEX COVER problem is polynomially solvable, such as bipartite graphs and line graphs.

Our hardness results remain valid for input instances in which the vertex requirements $r(v)$ are bounded by 4. On the other hand, we show that VEC CON can be solved in polynomial time for requirements bounded by 2, thus leaving open only the case with maximum requirement 3. We also develop a polynomial-time solution for VEC CON in block graphs, thereby generalizing the result by Boros et al. [4] showing that VEC CON is polynomial on trees. This result is obtained by introducing a more general problem called FREE-SET VECTOR CONNECTIVITY (FREEVEC CON for short), and developing an algorithm that reduces an instance of FREEVEC CON to solving instances of FREEVEC CON on biconnected components of the input graph.

Vertex covers and dominating sets in a graph G can be easily characterized as hitting sets of derived hypergraphs (of G itself, and of the closed neighborhood hypergraph of G , respectively). We give a similar characterization of vector connectivity sets.

The paper is structured as follows. In Section 1.1, we collect all the necessary definitions. In Section 2, we develop a characterization of vector connectivity sets as hitting sets of a derived hypergraph, which is followed with hardness results in Section 3. A polynomial reduction for a problem generalizing VEC CON to biconnected graphs is given in Section 4. Some of the algorithmic consequences of this reduction are examined in Section 5. We conclude the paper with some open problems in Section 6.

1.1. Definitions and Notation

All graphs in this paper are simple and undirected, and will be denoted by $G = (V, E)$, where V is the set of vertices and E is the set of edges. We use standard graph terminology. In particular, the degree of a vertex v in G is denoted by $d_G(v)$, the neighborhood and the closed neighborhood of a vertex v are denoted by $N_G(v)$ and $N_G[v]$, respectively, and $V(G)$ refers to the vertex set of G . Moreover, for a set $X \subseteq V(G)$, we define $N_G(X) = (\cup_{v \in X} N_G(v)) \setminus X$ and denote by $G[X]$ the subgraph of G induced by X . Given a graph $G = (V, E)$, a set $S \subseteq V$ and a vertex $v \in V \setminus S$, a v - S fan of order k is a collection of k paths P_1, \dots, P_k such that (1) every P_i is a path connecting v to a vertex of S , and (2) the paths are pairwise vertex-disjoint except at v , i.e., for all $1 \leq i < j \leq k$, it holds that $V(P_i) \cap V(P_j) = \{v\}$.

Given a graph $G = (V, E)$ and an integer-valued function $r : V \rightarrow \mathbb{Z}_+ = \{0, 1, 2, \dots\}$, a *vector connectivity set* for (G, r) is a set $S \subseteq V$ such that there exists a v - S fan of order $r(v)$ for every $v \in V \setminus S$. We say that $r(v)$ is the *requirement* of vertex v . The VEC CON problem is the problem of finding a vector connectivity set of minimum size for (G, r) . The minimum size of a vector connectivity set for (G, r) is denoted by $\kappa(G, r)$. In Boros et al. [4], it was assumed that vertex requirements do not exceed their degrees. Since our polynomial results are developed using the more general variant of the problem, we do not impose this restriction. At the same time, our hardness results also hold for the original, more restrictive variant.

For every $v \in V$ and every set $S \subseteq V \setminus \{v\}$, we say that v is r -linked to S if there is a v - S fan of order r in G . Hence, given an instance (G, r) of VEC CON, a set $S \subseteq V$ is a vector connectivity set for (G, r) if and only if every $v \in V \setminus S$ is $r(v)$ -linked to S . Given a vertex requirement function $r : V \rightarrow \mathbb{Z}$ and a non-empty set $X \subseteq V$, we define $R(X) := \max_{x \in X} r(x)$. A graph G is k -connected if $|V(G)| > k$ and for every $S \subseteq V(G)$ with $|S| < k$, the graph $G - S$ is connected.

2. A characterization of vector connectivity sets

Menger's Theorem [14] implies the following characterization of vector connectivity sets, showing that they are exactly the hitting sets of a certain hypergraph derived from graph G and vertex requirement function r . This characterization will be used in our proof of Theorem 1 in Section 3.

Proposition 1. *For every graph $G = (V, E)$, vertex requirements $r : V \rightarrow \mathbb{Z}_+$, and a set $S \subseteq V$, the following conditions are equivalent:*

- (i) *S is a vector connectivity set for (G, r) .*
- (ii) *For every non-empty set $X \subseteq V$ such that $G[X]$ is connected and $R(X) > |N_G(X)|$, we have $S \cap X \neq \emptyset$.*

Proof. First, let S be a vector connectivity set for (G, r) . Suppose for a contradiction that there exists a non-empty set $X \subseteq V$ such that $G[X]$ is connected, $R(X) > |N_G(X)|$, and $S \cap X = \emptyset$. Let $C = N_G(X)$, and let $x \in X$ be a vertex such that $r(x) > |C|$. Since $S \cap X = \emptyset$, we have $x \notin S$. Moreover, the definition of C implies that in the graph $G - C$, there is no path from x to S . Therefore, by Menger's Theorem, the maximum number of disjoint x - S paths is at most $|C|$, contrary to the fact that x is $r(x)$ -linked to S and $r(x) > |C|$.

Conversely, suppose for a contradiction that $S \subseteq V$ is not a vector connectivity set for (G, r) , and that for every non-empty set $X \subseteq V$ such that $G[X]$ is connected and $|N_G(X)| < R(X)$, we have $S \cap X \neq \emptyset$. Since S is not a vector connectivity set for (G, r) , there exists a vertex $x \in V \setminus S$ such that x is not $r(x)$ -linked to S . By Menger's Theorem, there exists a set $C \subseteq V \setminus \{x\}$ such that $|C| < r(x)$ and every path connecting x to S contains a vertex of C . Let X be the component of $G - C$ containing x . Then, $G[X]$ is connected and $N_G(X)$ is contained in C , implying $|N_G(X)| \leq |C| < r(x) \leq R(X)$. Hence, by the assumption on S , we have $S \cap X \neq \emptyset$. But this means that there exists a path connecting x to S avoiding C , contrary to the choice of C . \square

3. Hardness results

We start with the NP-hardness results.

Theorem 1. *The decision version of the VEC CON problem restricted to instances with maximum requirement 4 is NP-complete, even for:*

- *2-connected planar bipartite graphs of maximum degree 5 and girth at least k (for every fixed k),*
- *2-connected planar line graphs of maximum degree 5.*

Proof. Membership in NP follows from the fact that the feasibility of a solution S can be tested in polynomial time, using, e.g., Menger's Theorem and max flow algorithms.

We first show hardness of the problem for 2-connected planar line graphs of maximum degree 5, and then show how to modify the construction to obtain a 2-connected planar bipartite graph of maximum degree 5 and girth at least k (for a fixed k).

The hardness reduction is from VERTEX COVER in 2-connected cubic planar graphs, a problem shown NP-complete by Mohar [15]. Recall that in the VERTEX COVER problem, the input is a graph G and an integer k , and the task is to determine whether G contains a vertex cover of size at most k , where a *vertex cover* in a graph is a set of its vertices such that every edge of the graph has an endpoint in the set.

Suppose that (G, k) is an instance to the VERTEX COVER problem such that G is a 2-connected cubic planar graph.

We construct a graph $G' = (V', E')$ in 3 steps, using the following procedure:

1. Replace each edge $e = xy$ of G with a path on 5 vertices $(x, w_{x,e}, w_e, w_{y,e}, y)$. Formally, delete edge xy , add three new vertices $w_{x,e}, w_e, w_{y,e}$ and edges $xw_{x,e}, w_{x,e}w_e, w_e w_{y,e}, w_{y,e}y$.

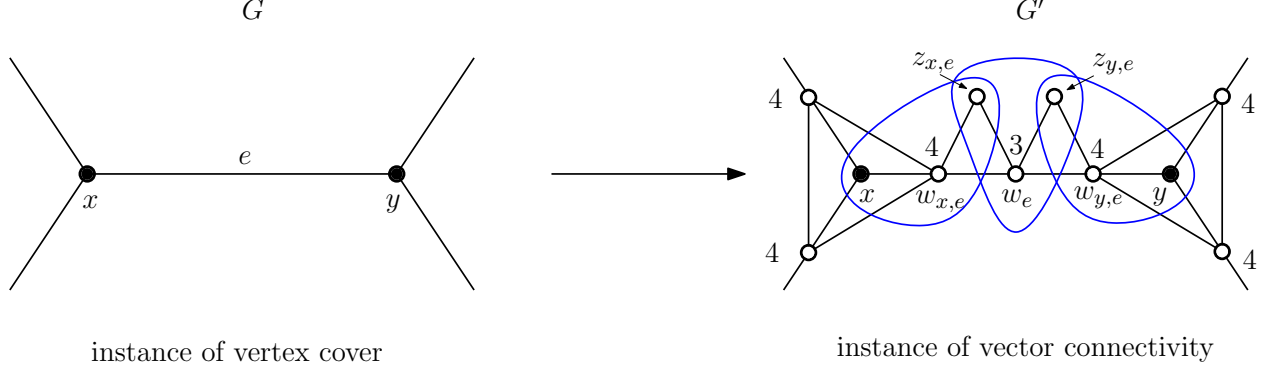


Figure 1: For each edge of G , every vector connectivity set for (G', r) must contain at least one vertex from each of the circled sets (see Claim 1); the three sets can be hit with two vertices, but not with just one. No matter whether one or two of the original (black) vertices are used, one of the additional (white) vertex must be used (and if no black vertex is used, two white vertices must be used). The best we can do is to find a vertex cover for G and then add one white vertex for each edge of G .

2. For each edge e of G , glue a triangle on top of $w_{x,e}w_e$ with tip $z_{x,e}$ (a new vertex), and a triangle on top of $w_{y,e}w_e$ with tip $z_{y,e}$ (a new vertex). Formally, add two new vertices $z_{x,e}$ and $z_{y,e}$ and edges $w_{x,e}z_{x,e}$, $z_{x,e}w_e$, $w_ez_{y,e}$, $z_{y,e}w_{y,e}$.
3. For every vertex x of G , let e, f, g be the three edges incident with x in G . Add the edges $w_{x,e}w_{x,f}$, $w_{x,e}w_{x,g}$, $w_{x,f}w_{x,g}$.

The obtained graph G' has $|V(G)| + 5|E(G)|$ vertices and $6|V(G)| + 6|E(G)|$ edges, and can be computed in polynomial time from G . Moreover, since a planar embedding of G can be easily transformed into a planar embedding of G' , we also have that G' is planar. Since G is 2-connected, so is G' , and G' is of maximum degree 5.

Furthermore, graph G' is a line graph. To see this, it suffices to observe that G' is isomorphic to the graph obtained from G with the following procedure:

1. Subdivide each edge of G twice. Let G_1 be the obtained graph.
2. Add to each vertex v of G_1 a private neighbor (that is, a vertex of degree 1 adjacent to v). Let G_2 be the obtained graph.
3. Take the line graph of G_2 .

To complete the reduction, we need to specify the requirements $r(\cdot)$ to vertices of G' . For every edge $e = xy$ of G , set $r(w_{x,e}) = r(w_{y,e}) = 4$, and $r(w_e) = 3$. Set $r(v') = 0$ to all other vertices v' of G' .

Let $\tau(G)$ denote the minimum size of a vertex cover of G . The NP-completeness of VEC CON with maximum requirement 4 in 2-connected planar line graphs of maximum degree 5 will follow from the following lemma. See Fig. 1 for a pictorial explanation of the reduction idea.

Lemma 1. $\kappa(G', r) = \tau(G) + |E(G)|$.

We will prove the lemma through a sequence of auxiliary statements. Then we will argue how to modify the construction to obtain a bipartite graph of arbitrarily high girth.

Claim 1: Let $S \subseteq V(G')$. Then, S is a vector connectivity set for (G', r) if and only if for every edge $e = xy$ of G , the set S contains at least one vertex from each of the sets $X_e := \{x, w_{x,e}, z_{x,e}\}$, $Z_e := \{z_{x,e}, w_e, z_{y,e}\}$, $Y_e := \{z_{y,e}, w_{y,e}, y\}$.

Proof of Claim 1: Let e be an edge of G . Each of the sets X_e, Y_e , and Z_e induces a connected subgraph in G' . Moreover $|N_{G'}(X_e)| = |N_{G'}(Z_e)| = 3 < 4 = R(X_e) = R(Z_e)$, and $|N_{G'}(Y_e)| = 2 < 3 = R(Y_e)$. Thus, if S is a vector connectivity set for (G', r) , then Proposition 1 implies that S contains a vertex from each of the sets X_e, Y_e , and Z_e .

Conversely, suppose that for every edge e of G , set S contains at least one vertex from each of the sets X_e, Y_e , and Z_e . Let v' be a vertex in $V(G') \setminus S$. Trivially, if $r(v') = 0$ then v' is $r(v')$ -linked to S .

Suppose that $r(v') = 3$. Then $v' = w_e$ for some edge $e = xy$ of G . Since S contains a vertex from Z_e , we may assume w.l.o.g. that $z_{x,e} \in S$. Let $e' = xy'$ be an edge incident with x other than e . Using the assumption on S , we can now find 3 internally vertex disjoint paths linking v' to S with respective endpoints x_1, x_2, x_3 such that $x_1 \in Z_{e'} \cup \{w_{x,e}\}$, $x_2 = z_{x,e}$ and $x_3 \in Y_{e'}$.

Suppose that $r(v') = 4$. Then, up to symmetry, $v' = w_{x,e}$ for some edge $e = xy$ of G . Let $e' = xy'$ and $e'' = xy''$ be the two edges incident with x other than e . Using the assumption on S , we can now find 4 internally vertex disjoint paths linking v' to S with respective endpoints x_1, \dots, x_4 such that $x_1 \in X_{e'} \cup Z_{e'}$, $x_2 \in X_{e''} \cup Z_{e''}$, $x_3 \in \{x, z_{x,e}\}$ and $x_4 \in \{w_e\} \cup Y_e$.

Since every vertex $v' \in V(G') \setminus S$ is $r(v')$ -linked to S , we conclude that S is a vector connectivity set for (G', r) .

Claim 2: $\kappa(G', r) \leq \tau(G) + |E(G)|$.

Proof of Claim 2: Let C be an optimal vertex cover of G . First, we extend C to a vertex cover C' of graph G_1 (recall that G_1 is the graph obtained from G by a double subdivision of each edge). This can be done by adding exactly one additional vertex for each edge of G . Hence, $|C'| \leq |C| + |E(G)|$. Now, consider the set $S \subseteq V(G')$ obtained as follows:

- (1) Put in S all vertices of C .
- (2) For every edge $e = xy$ of G , label the vertices of the 4-vertex path replacing e in G_1 as $(x, z'_{x,e}, z'_{y,e}, y)$. Since $z'_{x,e}z'_{y,e} \in E(G_1)$ and C' is a vertex cover of G_1 , we have C' contains either $z'_{x,e}$ or $z'_{y,e}$. If $z'_{x,e} \in C'$, then add $z_{x,e}$ to S . Otherwise, add $z_{y,e}$ to S .

It can be verified using Claim 1 that S is a vector connectivity set for (G', r) of total size at most $\tau(G) + |E(G)|$. This completes the proof of the claim.

For a subset S of $V(G')$, let $n(S)$ denote the total number of elements of the form $w_{x,e}, w_e, w_{y,e}$ contained in S . Equivalently, this is the number of non-simplicial vertices of G' contained in S . (A vertex in a graph is said to be *simplicial* if its neighborhood forms a clique. The closed neighborhood of a simplicial vertex v is said to be a *simplicial clique (rooted at v)*.)

Claim 3: There is a minimum vector connectivity set for (G', r) with $n(S) = 0$.

Proof of Claim 3: Let S be a minimum vector connectivity set for (G', r) that minimizes $n(S)$. Suppose for a contradiction that $n(S) > 0$. By symmetry, we may assume that there exists a vertex $w \in S$ such that $w \in \{w_{x,e}, w_e\}$ for some edge $e = xy$ of G . By minimality of S and Claim 1, set S does not contain $z_{x,e}$ (otherwise, $S \setminus \{w\}$ would be a vector connectivity set smaller than S). Let $S' = (S \setminus \{w\}) \cup \{z_{x,e}\}$. By Claim 1, S' is a vector connectivity set for (G', r) . Since $|S'| \leq |S|$, set S' is a minimum vector connectivity set. However, we have $n(S') < n(S)$, contradicting the choice of S .

We are now ready to complete the proof of Lemma 1.

Proof of Lemma 1: Claim 1 established that $\kappa(G', r) \leq \tau(G) + |E(G)|$.

The proof of the reverse inequality can be derived from Claims 2 and 3, as follows. By Claim 3, there exists a minimum vector connectivity set S for (G', r) with $n(S) = 0$. By Claim 2 and since $n(S) = 0$, set S has the following property: for every edge $e = xy$ of G , set S contains at least one vertex from each of the sets $\{x, z_{x,e}\}, \{z_{x,e}, z_{y,e}\}, \{z_{y,e}, y\}$. Assuming that for every edge $e = xy$ of G , the vertices of the 4-vertex path replacing e in G_1 are labeled as $(x, z'_{x,e}, z'_{y,e}, y)$, the above property implies that the set obtained from S by replacing each vertex of the form $z_{x,e}$ with $z'_{x,e}$, and each vertex of the form $z_{y,e}$ with $z'_{y,e}$, is a vertex

cover of G_1 . Therefore, the vertex cover number of G_1 is at most $|S| = \kappa(G', r)$. Since the vertex cover numbers of G and G_1 are related with the equality $\tau(G) = \tau(G_1) - |E(G)|$ (see, e.g., [17]), it follows that $\tau(G) = \tau(G_1) - |E(G)| \leq \kappa(G', r) - |E(G)|$, as desired.

It remains to show how to modify the construction to obtain a bipartite graph of girth at least k (while keeping planarity, 2-connectivity and maximum degree). We take the graph G' and subdivide each edge $2k + 1$ times. Since we subdivided each edge an odd number of times, the obtained graph \tilde{G} is bipartite. Clearly, girth of \tilde{G} is more than k , while planarity, 2-connectivity and maximum degree are maintained. On original vertices, we keep the requirements as above. To each new vertex, we assign requirement 0. Let $\tilde{r} : V(\tilde{G}) \rightarrow \{0, 3, 4\}$ be the new requirement function. In this setting, equality $\kappa(\tilde{G}, \tilde{r}) = \tau(G) + |E(G)|$ holds, which will establish NP-completeness.

The inequality \leq can be proved similarly as the inequality $\kappa(G', r) \leq \tau(G) + |E(G)|$ above in Claim 1. For the proof of the reverse inequality, we need to modify Claims 1 and 3 appropriately. First, observe that each edge of G' is contained in a unique simplicial clique (rooted at a vertex of the form x or $z_{x,e}$ where $x \in V(G)$ and $e \in E(G)$). Hence, we can associate to each edge e' of G' the simplicial vertex $s(e')$ of G' such that $e' \subseteq N_{G'}[s(e')]$. For a simplicial vertex s of G' , let $A(s) = \{s\} \cup B(s)$, where $B(s)$ is the set of all vertices in \tilde{G} that were introduced on some edge e' of G' with $s(e') = s$ in the process of constructing \tilde{G} from G' .

In this setting, Claim 1 is replaced with the following claim.

Claim 1': Let \tilde{S} be a subset of $V(\tilde{G})$. Then, \tilde{S} is a vector connectivity set for (\tilde{G}, \tilde{r}) if and only if for every edge $e = xy$ of G , set \tilde{S} contains at least one vertex from each of the sets $A(x) \cup \{w_{x,e}\} \cup A(z_{x,e})$, $A(z_{x,e}) \cup \{w_e\} \cup A(z_{y,e})$, $A(z_{y,e}) \cup \{w_{y,e}\} \cup A(y)$.

The proof is similar to that of Claim 1.

For a subset \tilde{S} of $V(\tilde{G})$, let $n(\tilde{S})$ denote the total number of non-simplicial vertices of \tilde{G} contained in \tilde{S} . Claim 3 is replaced with the following claim.

Claim 3': There is a minimum vector connectivity set for (\tilde{G}, \tilde{r}) with $n(\tilde{S}) = 0$.

The proof is similar to that of Claim 3. Vertices of the form $x \in V(G)$ and w_e for $e \in E(G)$ are handled similarly as in the proof of Claim 2. If \tilde{S} contains a non-simplicial vertex w of degree 2, we can simply replace it with vertex s , where s is the simplicial vertex of \tilde{G} such that $w \in A(s)$. \square

A similar approach as one used to prove Theorem 1 can be used to show the following inapproximability result. Recall that APX is the class of problems approximable in polynomial time to within some constant, and that a problem Π is APX-hard if every problem in APX reduces to Π via an AP-reduction [3]. APX-hard problems do not admit a polynomial-time approximation scheme (PTAS), unless $P = NP$. To show that a problem is APX-hard, it suffices to show that an APX-complete problem is L-reducible to it [3]. Given two NP optimization problems Π and Π' , we say that Π is L-reducible to Π' if there exists a polynomial-time transformation f from instances of Π to instances of Π' and positive constants α and β such that for every instance x of Π , we have:

1. $\text{opt}_{\Pi'}(f(x)) \leq \alpha \cdot \text{opt}_{\Pi}(x)$, and
2. for every feasible solution y' of $f(x)$ with objective value c_2 we can compute in polynomial time a solution y of x with objective value c_1 such that $|\text{opt}_{\Pi}(x) - c_1| \leq \beta \cdot |\text{opt}_{\Pi'}(f(x)) - c_2|$.

Theorem 2. *VECCON is APX-hard. In particular, VEC CON admits no PTAS, unless $P = NP$.*

Proof. Since VERTEX COVER is APX-complete for cubic graphs [2], it suffices to show that VERTEX COVER in cubic graphs is L-reducible to VEC CON. Consider the polynomial-time transformation described in the first part of the proof of Theorem 1, that starts from a cubic graph G (not necessarily planar or 2-connected), an instance to VERTEX COVER, and computes an instance (G', r) of VEC CON.

By Lemma 1, we have $\kappa(G', r) = \tau(G) + |E(G)|$. Moreover, since G is cubic, every vertex in a vertex cover of G covers exactly 3 edges, hence $\tau(G) \geq \frac{|E(G)|}{3}$. This implies that $\kappa(G', r) = \tau(G) + |E(G)| \leq 4\tau(G)$,

hence the first condition in the definition of L -reducibility is satisfied with $\alpha = 4$. The second condition in the definition of L -reducibility states that for every vector connectivity set S of (G', r) we can compute in polynomial time a vertex cover C of G such that $|C| - \tau(G) \leq \beta \cdot |S| - \kappa(G, r)$ for some $\beta > 0$. We claim that this can be achieved with $\beta = 1$. Indeed, the proof of Lemma 1 above shows how one can transform in polynomial time any vector connectivity set S in G' to a vertex cover C of G such that $|C| \leq |S| - |E(G)|$. Therefore, $|C| - \tau(G) \leq |S| - |E(G)| - \tau(G) = |S| - \kappa(G', r)$. This shows that VERTEX COVER in cubic graphs is L -reducible to VEC CON, and completes the proof. \square

4. Reduction to biconnected graphs

A *cut vertex* in a connected graph G is a vertex v such that $G - v$ is disconnected. A graph is *biconnected* if it is connected and has no cut vertices. A *block* (or a *biconnected component*) of a connected graph G is a maximal biconnected subgraph of G . The blocks of a connected graph G are connected in a tree structure, which is called the *block tree* of G , and whose leaves are referred to as the *leaf blocks* of G . The block tree of a given connected graph can be computed in linear time.

Boros et al. [4] proved that VEC CON is polynomial for trees. In this section, we present an algorithm showing that VEC CON is polynomial also for the larger class of *block graphs*, that is, graphs every block of which is complete. Our result will follow from a solution to a more general problem on block graphs, which we call FREE-SET VECTOR CONNECTIVITY (FREEVEC CON). In this problem, a subset F of *free vertices* is also given as part of the input, and the requirement that S is a vector connectivity set is relaxed to the requirement that every vertex in $V \setminus S$ is $r(v)$ -linked to $S \cup F$. When $F = \emptyset$, the problem is equivalent to VEC CON. Formally, given a graph $G = (V, E)$, an integer-valued function $r : V \rightarrow \mathbb{Z}_+$ and a subset $F \subseteq V$, a *vector connectivity set* for (G, F, r) is a set $S \subseteq V$ such that every vertex in $V \setminus S$ is $r(v)$ -linked to $S \cup F$. The FREEVEC CON problem is the problem of finding a vector connectivity set of minimum size for (G, F, r) . Given a graph G , a vertex $v \in V(G)$ and a subset $S \subseteq V(G)$, we will denote by $\kappa_G(v, S)$ the maximum k such that v is k -linked to S .

We present an algorithm with which we can show that if for a given hereditary class of graphs \mathcal{G} , the FREEVEC CON problem can be solved in polynomial time on biconnected graphs in \mathcal{G} then the FREEVEC CON problem is also solvable in polynomial time on graphs from \mathcal{G} . Clearly, when solving the VEC CON and FREEVEC CON problems we may restrict our attention to connected graphs.

Theorem 3. *Suppose that there exists an $\mathcal{O}(T)$ algorithm for the FREEVEC CON problem on biconnected graphs from a class \mathcal{G} . Then, the FREEVEC CON problem on a connected graph G every block of which is in \mathcal{G} can be solved in time*

$$\mathcal{O}\left(|V(G)|^2 \min\left\{r_{\max}, |V(G)|^{1/2}\right\} \min\{r_{\max}|V(G)|, |E(G)|\} + |V(G)|r_{\max}T\right), \quad (1)$$

where $r_{\max} = \max\{2, \max\{r(v) \mid v \in V(G)\}\}$.

Proof. Assume that FSVecCon-biconnect is an algorithm that in polynomial time $\mathcal{O}(T)$ correctly solves the FREE VECTOR CONNECTIVITY problem on instances (G, F, r) such that $G \in \mathcal{G}$ is biconnected and $r : V(G) \rightarrow \{0, 1, \dots, r_{\max}\}$. We claim that **Procedure FSVecCon** correctly solves FREEVEC CON in time given by (1).

We will first prove the correctness and optimality of the solution returned by FSVecCon. At the end, we will analyze the time complexity of the algorithm.

Lines 1–3. If G is biconnected, then in lines 1–3, the solution is constructed using procedure FSVecCon-biconnect. In this case, feasibility and optimality follow from the assumption on FSVecCon-biconnect which is a polynomial time algorithm for FREEVEC CON restricted to instances (G, F, r) such that G is biconnected and $r : V(G) \rightarrow \mathbb{Z}$.

If G is not biconnected, then in line 4, the algorithm checks if the trivial solution works, and if not, then in lines 5–8, the algorithm identifies a leaf block B and computes the Boolean values β and ρ . These values

Algorithm 1 Reduces FREEVECCON to the biconnected case

Procedure FSVecCon(G, F, r)**Input:** A connected graph $G = (V, E)$, a subset $F \subseteq V$, a function $r : V \rightarrow \mathbb{Z}_+$.**Output:** A minimum vector connectivity set for (G, F, r) .

```
1: if  $G$  is biconnected then
2:   return FSVecCon-biconnect( $G, F, r$ )
3: end if

4: if  $\emptyset$  is a vector connectivity set for  $(G, F, r)$  then return  $\emptyset$ 

5: Let  $B$  be a leaf block of  $G$ ;  $v$  the cut vertex contained in  $B$ ;  $R \leftarrow G - (V(B) \setminus \{v\})$ ;
6: Let  $\beta$  and  $\rho$  be two Boolean values computed as follows:
7:    $\beta = \text{true}$  iff every vertex  $u \in V(B) \setminus \{v\}$  is  $r(u)$ -linked to  $F \cup \{v\}$ 
8:    $\rho = \text{true}$  iff every vertex  $w \in V(R) \setminus \{v\}$  is  $r(w)$ -linked to  $F \cup \{v\}$ ;
9: if  $\beta \wedge \rho = \text{true}$  then return  $\{v\}$ 

10: if  $\beta \wedge \rho = \text{false}$  and  $\beta \vee \rho = \text{true}$  (that is, exactly one between  $\beta$  and  $\rho$  is true) then
11:   if  $\beta = \text{true}$  then  $H \leftarrow R$  else  $H \leftarrow B$ 
12:   if  $F \not\subseteq V(H) \setminus \{v\}$  then
13:      $r(v) \leftarrow \max\{r(v) - \kappa_G(v, F \setminus V(H)) + |F|, 0\}$ 
14:      $F \leftarrow F \cup \{v\}$ 
15:   end if
16:   return FSVecCon( $H, F \cap V(H), r|_{V(H)}$ )
17: end if

18: if  $\beta \vee \rho = \text{false}$  then
19:   for  $i \in \{0, 1, \dots, r(v)\}$  do
20:      $r(v) \leftarrow i$ ;  $S_i \leftarrow \text{FSVecCon-biconnect}(B, (F \cap V(B)) \cup \{v\}, r|_{V(B)})$ 
21:   end for
22:    $i^* = \max\{j \mid 0 \leq j \leq r(v), |S_j| = |S_0|\}$ 
23:    $r(v) \leftarrow r(v) - i^* + 2$ 
24:    $S_R \leftarrow \text{FSVecCon}(R, (F \cap V(R)) \cup \{v\}, r|_{V(R)})$ 
25:   return  $S_{i^*} \cup S_R$ 
26: end if
```

are meant to indicate whether the set of free vertices together with the cut vertex satisfy the connectivity requirements in the leaf block (β) and in the remaining part of the graph (ρ), respectively.

More precisely, let B be a leaf block of G and let $v \in V(B)$ be the cut vertex of G contained in B . Let $R = G - (V(B) \setminus \{v\})$. The algorithm computes the values

$$\beta = \begin{cases} \text{true}, & \text{if every vertex } u \in V(B) \setminus \{v\} \text{ is } r(u)\text{-linked to } F \cup \{v\}; \\ \text{false}, & \text{otherwise.} \end{cases}$$

and

$$\rho = \begin{cases} \text{true}, & \text{if every vertex } w \in V(R) \setminus \{v\} \text{ is } r(w)\text{-linked to } F \cup \{v\}; \\ \text{false}, & \text{otherwise.} \end{cases}$$

The remaining part of the computation is based on the values β and ρ and may imply recursive calls of the algorithm. In the analysis below, whenever the meaning will be clear from the context, we will not distinguish between a graph and its vertex set. Let OPT denote the minimum size of a vector connectivity set for (G, F, r) .

Case A. $\beta = \rho = \text{true}$.

This together with the fact that the empty set is not a solution implies that $OPT = 1$. In fact, the assumption that $\beta = \rho = \text{true}$ implies that $\{v\}$ is a vector connectivity set for (G, F, r) .

Case B. Exactly one of the values β and ρ is true.

This means that the cut vertex v and F can satisfy the requirements of only one of the two parts of the graph on the two sides of the cut vertex. The algorithm is then recursively invoked on the part of the graph in which the requirements are not satisfied. In the following we show that the solution so computed is indeed a solution for the original problem. Since the two cases are similar to each other, we shall limit ourselves to discuss only the case where $\beta = \text{true}$ and $\rho = \text{false}$. Under this assumption, we have $H = R$, and the condition $F \not\subseteq V(H) \setminus \{v\}$ in line 12 is equivalent to the condition $F \cap B \neq \emptyset$.

Observation 1. Since $\rho = \text{false}$, there exists a vertex $w \in V(R) \setminus \{v\}$ that is not $r(w)$ -linked to $F \cup \{v\}$. In particular, w is not $r(w)$ -linked to F , therefore any solution S to the FREEVECCON problem on $(R, F \cap R, r')$, with $r'(w) = r(w)$ for each $w \neq v$, is non-empty. The same holds true for the solutions to the problem on $(R, (F \cap R) \cup \{v\}, r')$,

Observation 2. There exists a minimum vector connectivity set S^* for (G, F, r) such that $S^* \subseteq R$. Indeed, if S^* is a minimum vector connectivity set for (G, F, r) and $u \in S^* \setminus R$, then $(S^* \setminus \{u\}) \cup \{v\}$ is a vector connectivity set for (G, F, r) .

We shall split the analysis into two cases according to the intersection of $F \cap B$.

Subcase B-1. $F \cap B = \emptyset$.

Then the algorithm is called recursively on $(R, F, r|_R)$ and the solution obtained is returned.

Feasibility. Since $\beta = \text{true}$, every vertex $u \in B \setminus \{v\}$ is $r(u)$ -linked to $F \cup \{v\}$. Since $F \cap B = \emptyset$ and v is a cut vertex, every vertex $u \in B \setminus \{v\}$ is $r(u)$ -linked to $\{v\}$, hence $r(u) \leq 1$. On the other hand, if S is the solution returned, which is non-empty by Observation 1, every vertex $u \in B \setminus \{v\}$ is $r(u)$ -linked to S (and hence to $S \cup F$). Since S is a vector connectivity set for $(R, F, r|_R)$, we have that S is also a vector connectivity set for (G, F, r) , which establishes feasibility.

Optimality. Let S^* be a minimum vector connectivity set for (G, F, r) such that $S^* \subseteq R$ (such a set exists by Observation 2). But now, every vertex $w \in R \setminus S^*$ is $r(w)$ -linked to $S^* \cup F$. Since $S^* \cup F \subseteq R$, every path connecting w to $S^* \cup F$ lies entirely in R . Thus, S^* is a vector connectivity set for $(R, F, r|_R)$. By the choice of S , we have $|S| \leq |S^*|$, which establishes optimality.

Subcase B-2. $F \cap B \neq \emptyset$. In this case, we define a new requirement assignment for the vertices in R as follows. Let $r' : V(R) \rightarrow \{0, 1, \dots, r_{\max}\}$ be defined as

$$r'(w) = \begin{cases} \max\{r(v) - \kappa_G(v, F \cap (B \setminus \{v\})) + |\{v\} \setminus F|, 0\} & \text{if } w = v; \\ r(w), & \text{otherwise.} \end{cases}$$

for all $w \in V(R)$. Then, the algorithm is recursively executed on $(R, (F \cap R) \cup \{v\}, r')$ and the obtained solution is returned.

Feasibility. Let S be the solution returned in this step. By definition, and Observation 1 we have $\emptyset \neq S \subseteq R$, and every vertex $w \in R \setminus S$ is $r'(w)$ -linked to $(S \cup (F \cap R)) \cup \{v\}$ (in R). To establish feasibility, we need to verify that every vertex $w \in G \setminus S$ is $r(w)$ -linked to $S \cup F$ (in G). If $w \in R \setminus \{v\}$, then the condition holds due to above property of S and the fact that $r(w) = r'(w)$. If $w \in B \setminus \{v\}$, by the assumption that $\beta = \text{true}$, it follows that w is $r(w)$ -linked to $F \cup \{v\}$. Now, if $v \in F$, it immediately follows that w is $r(w)$ -linked to $S \cup F$. On the other hand, if $v \notin F$, it is easy to see that w is $r(w)$ -linked to $S \cup F$ because of $\emptyset \neq S \subseteq R$ since we can extend the path to v to a path to S .

Finally, suppose that $w = v$. Let $k = \kappa_G(v, F \cap (B \setminus \{v\}))$. Vertex v is k -linked to $F \cap (B \setminus \{v\})$ in G (equivalently: in B). If $k \geq r(v)$ then v is also $r(v)$ -linked (in G) to $S \cup F$. If $k < r(v)$, then $r(v) - k + |\{v\} \setminus F| \geq 0$, and v is also $(r(v) - k + |\{v\} \setminus F|)$ -linked in R to $(S \cup (F \cap R)) \cup \{v\}$. This immediately implies that: (i) if $v \in F$ then v is $r(v)$ -linked (in G) to $S \cup F$; if $v \notin F$ then v is $(r(v) + 1)$ -linked to $S \cup F \cup \{v\}$, hence it is $r(v)$ -linked (in G) to $S \cup F$.

Optimality. Let S^* be a minimum vector connectivity set for (G, F, r) such that $S^* \subseteq R$ (such a set exists by Observation 2). It suffices to show that S^* is a vector connectivity set for $(R, (F \cap R) \cup \{v\}, r')$, since then we will have $|S| \leq |S^*|$, establishing optimality. Let $w \in R \setminus S^*$. Since S^* is a vector connectivity set

for (G, F, r) , vertex w is $r(w)$ -linked to $S^* \cup F$ in G . If $w \neq v$, then $r(w) = r'(w)$, hence w is $r'(w)$ -linked to $S^* \cup F$ in G . Since v is a cut vertex, this implies that w is $r'(w)$ -linked to $S^* \cup (F \cap R) \cup \{v\}$ in R . Suppose now that $w = v$. If $r'(v) = 0$, then v is trivially $r'(v)$ -linked in to $S^* \cup (F \cap R) \cup \{v\}$ in R . If $r'(v) > 0$, then $r'(v) = r(v) - \kappa_G(v, F \cap (B \setminus \{v\})) + |\{v\} \setminus F|$, and v is $r(v)$ -linked to $S^* \cup F$ in G . Therefore

$$\begin{aligned}
& \kappa_G(v, F \cap (B \setminus \{v\})) + \kappa_R(v, S^* \cup (F \cap R) \cup \{v\}) \\
&= \kappa_G(v, S^* \cup F \cup \{v\}) \\
&= \kappa_G(v, S^* \cup F) + |\{v\} \setminus F| \\
&\geq r(v) + |\{v\} \setminus F| \\
&= \kappa_G(v, F \cap (B \setminus \{v\})) + r'(v),
\end{aligned}$$

and v is again $r'(v)$ -linked to $S^* \cup (F \cap R) \cup \{v\}$ in R .

Case C. $\beta = \rho = \text{false}$.

The idea is to first compute a solution S_B for the block B that does not include vertex v and which maximizes the value k such that v is k -linked to $S_B \cup (F \cap B)$. To this aim, we compute the sets S_i , for $i = 0, \dots, r(v)$, defined as the solution to the instance $(B, (F \cap B) \cup \{v\}, r_i)$, where r_i is the restriction of r to B , modified so that the requirement of v is $r_i(v) = i$.

In formulae, for all $i \in \{0, 1, \dots, r(v)\}$, let r_i be a new requirement assignment defined as

$$r_i(u) = \begin{cases} i, & \text{if } u = v; \\ r(u), & \text{otherwise.} \end{cases}$$

Using algorithm **FSVecCon-biconnect**, compute an optimal solution S_i to the **FREEVECCON** problem on $(B, (F \cap V(B)) \cup \{v\}, r_i)$. Let i^* be the maximum in $\{0, 1, \dots, r(v)\}$ such that $|S_{i^*}| = |S_0|$.

Observation 3. $v \notin S_{i^*}$. Indeed, the set $S = S_{i^*} \setminus \{v\}$ is a vector connectivity set for $(B, (F \cap B) \cup \{v\}, r_0)$. Thus $|S_0| \leq |S| \leq |S_{i^*}| = |S_0|$, which implies that $|S| = |S_{i^*}|$ and consequently $v \notin S_{i^*}$. In particular, the set $T = S_{i^*} \cup (F \cap B) \cup \{v\}$ satisfies all the requirements in $B \setminus \{v\}$ and since $v \notin S_{i^*}$, vertex v is i^* -linked to T .

We set $S_B = S_{i^*}$.

Feasibility. First, let us argue that we have $r(v) - i^* + 2 \leq r_{\max}$, thus the instance $(R, (F \cap R) \cup \{v\}, r|_R)$ from the recursive call in line 24 has requirements at most r_{\max} . Indeed, suppose for a contradiction that $r(v) - i^* + 2 \geq r_{\max} + 1$. This implies that $r(v) \geq i^* + r_{\max} + 1$. Since $r(v) \geq 1$, by the definition of i^* it also follows that $i^* \geq 1$. Hence, it must be $r(v) = r_{\max}$ and $i^* = 1$. By definition of i^* , vertex v is not 2-linked to $(S_0 \cup (F \cap B)) \cup \{v\}$, in particular, $S_0 = \emptyset$. Consequently, every vertex $u \in B \setminus \{v\}$ is $r(u)$ -linked to $S_0 \cup (F \cap B) \cup \{v\} \subseteq F \cup \{v\}$, which is a contradiction to the assumption that $\beta = \text{false}$.

Since β and ρ are both **false**, it follows that $|S_B \cap (B \setminus \{v\})| \geq 1$ and $|S_R \cap (R \setminus \{v\})| \geq 1$. Therefore, from S_R (resp. S_B) there is a path to v . Each $w \in (B \setminus S_B) \setminus \{v\}$ (resp. $w \in (R \setminus S_R) \setminus \{v\}$) is $r(w)$ -linked to $S_B \cup (F \cap B) \cup \{v\}$ (resp. $S_R \cup (F \cap R) \cup \{v\}$). Therefore, if one of such paths is to v , it can be extended to a path to S_R (reps. S_B). Hence w is $r(w)$ -linked to $S_B \cup S_R \cup F$, as required.

As for v , by the definition of S_R and S_B we have that v is at least $(i^* - 1)$ -linked to S_B and at least $(r(v) - i^* + 1)$ -linked to $S_R \cup (F \cap (R \setminus \{v\}))$. This follows from having required of S_B and S_R that v is i^* -linked to $S_B \cup (F \cap B) \cup \{v\}$, and v is $(r(v) - i^* + 2)$ -linked to $S_R \cup (F \cap R) \cup \{v\}$. Altogether we have $\kappa_G(v, S_B \cup S_R \cup F) \geq r(v)$, as required.

Optimality. Recall that $v \notin S_B = S_{i^*}$ (by Observation 3). Let S^* be an optimal solution. Because of $\rho = \text{false}$, we have $|S^* \cap (R \setminus \{v\})| \geq 1$. Analogously, $\beta = \text{false}$ implies $|S^* \cap (B \setminus \{v\})| \geq 1$. Furthermore, since $S^* \cap (B \setminus \{v\})$ is a vector connectivity set for $(B, (F \cap B) \cup \{v\}, r_0)$, we have $|S^* \cap (B \setminus \{v\})| \geq |S_0| = |S_B|$.

We now argue according to several cases, depending on whether $v \in S^*$ or not and depending on the value of i^* .

Case 1. $v \in S^*$.

In this case, the set $S^* \cap R$ is a feasible solution to the instance $(R, (F \cap R) \cup \{v\}, r')$, where $r'(w) = r(w)$ for each $w \in R \setminus \{v\}$ and $r'(v) = r(v) - i^* + 2$. Hence, $|S^* \cap R| \geq |S_R|$ and we have

$$|S^*| \geq |S^* \cap (B \setminus \{v\})| + |S^* \cap R| \geq |S_B| + |S_R| = |S_B \cup S_R|$$

where the last equality follows by $S_B \cap S_R \subseteq \{v\} \not\subseteq S_B$ (by Observation 3).

Case 2. $v \notin S^$.*

Subcase 2.1. $i^ = r(v)$.* In this case we have $|S_0| = |S_1| = \dots = |S_{r(v)}|$.

We have $S_B = S_{r(v)}$ and S_R is the optimal solution to the instance $(R, (F \cap R) \cup \{v\}, r')$, where $r'(w) = r(w)$ for each $w \neq v$ and $r(v) = 2$. Note that the set $S^* \cap R$ is a feasible solution for the instance $(R, (F \cap R) \cup \{v\}, r')$. This easily follows from $S^* \cap R = S^* \cap (R \setminus \{v\}) \neq \emptyset$, which guarantees the fact that v is at least 2-linked to $S^* \cup ((F \cap R) \cup \{v\})$. Therefore, we have $|S^* \cap R| \geq |S_R|$. Altogether we get

$$|S^*| \geq |S^* \cap (B \setminus \{v\})| + |S^* \cap R| \geq |S_B| + |S_R| = |S_B \cup S_R|.$$

Subcase 2.2. $i^ < r(v)$.* In this case we have $|S_0| = |S_1| = \dots = |S_{i^*}| < |S_{i^*+1}|$.

Let $k = \kappa_G(v, (S^* \cup F \cup \{v\}) \cap B)$.

Suppose first that $k \geq i^* + 1$. Then the set $S^* \cap B$ is a feasible solution for the instance $(B, (F \cap B) \cup \{v\}, r_{i^*+1})$, and hence $|S^* \cap B| \geq |S_{i^*+1}| \geq |S_{i^*}| + 1 = |S_B| + 1$. Moreover, the set $(S^* \cap R) \cup \{v\}$ is a solution to the instance $(R, (F \cap R) \cup \{v\}, r')$, where $r'(w) = r(w)$ for each $w \in R \setminus \{v\}$ and $r'(v) = r(v) - i^* + 2$. Hence, $|(S^* \cap R) \cup \{v\}| \geq |S_R|$, that is, $|S^* \cap R| \geq |S_R| - 1$. Altogether we get

$$|S^*| \geq |S^* \cap B| + |S^* \cap R| \geq |S_B| + 1 + |S_R| - 1 = |S_B \cup S_R|.$$

Suppose now that $k \leq i^*$. In this case, since

$$k + \kappa_R(v, (S^* \cup F) \cap R) = \kappa_G(v, S^* \cup F \cup \{v\}) \geq r(v) + 1,$$

we get $\kappa_R(v, (S^* \cup F) \cap R) \geq r(v) - i^* + 1$. Hence, v is $(r(v) - i^* + 2)$ -linked to $(S^* \cap R) \cup (F \cap R) \cup \{v\}$, and the set $S^* \cap R$ is a solution to the instance $(R, (F \cap R) \cup \{v\}, r')$, where $r'(w) = r(w)$ for each $w \in R \setminus \{v\}$ and $r'(v) = r(v) - i^* + 2$. Hence, $|S^* \cap R| \geq |S_R|$. Recall that $|S^* \cap (B \setminus \{v\})| \geq |S_B|$, and so altogether we get

$$|S^*| \geq |S^* \cap (B \setminus \{v\})| + |S^* \cap R| \geq |S_B| + |S_R| = |S_B \cup S_R|.$$

This establishes the correctness of the algorithm. It remains to analyze its time complexity.

We need a single call to the procedure **FSVecCon-biconnect**. Given a graph H , a set $U \subseteq V(H)$, a vertex $v \in V(H)$ and an integer k , it can be checked whether vertex v is k -linked to U in time

$$\mathcal{O}\left(\min\left\{k^2|V(H)|, k|V(H)|^{3/2}, k|E(H)|, |V(H)|^{1/2}|E(H)|\right\}\right) \quad (2)$$

The first two terms in the minimum come from the result of Nagamochi and Ibaraki [16]. The last two terms come from the result of Even and Tarjan [9].

Therefore, using the fact that $r(v) \leq r_{\max}$ for all $v \in V$, the Boolean values β and ρ in lines 6–8 can be computed in time

$$\mathcal{O}(|V(G)| \min\{r_{\max}, |V(G)|^{1/2}\} \min\{r_{\max}|V(G)|, |E(G)|\}).$$

The same time complexity can be similarly shown for line 9. For lines 10–17, observe that lines 10 and 11 take $\mathcal{O}(1)$ time and line 12 takes $\mathcal{O}(|V(G)|)$ time. Line 13 can also be implemented in

$$\mathcal{O}(r_{\max} \min\{r_{\max}, |V(G)|^{1/2}\} \min\{r_{\max}|V(G)|, |E(G)|\})$$

time since it suffices to check whether $\kappa_G(v, F \setminus V(H)) \geq i$ for each $i \in \{1, \dots, r(v) + |\{v\} \setminus F|\} \subseteq \{1, \dots, r_{\max} + 1\}$, and we can use the bound in (2). In line 16, one recursive call to the procedure is made. Lines 18–26 require one recursive call to the procedure, while the remaining steps in these lines can be implemented in time $\mathcal{O}(|V(G)|)$.

In either case, at most one recursive call to the algorithm is made, and the rest takes $\mathcal{O}(|V(G)| \min\{r_{\max}, |V(G)|^{1/2}\} \min\{r_{\max}|V(G)|, |E(G)|\})$ time. Since there are at most $\mathcal{O}(|V(G)|)$ recursive calls, this results in a total complexity of $\mathcal{O}(|V(G)|^2 \min\{r_{\max}, |V(G)|^{1/2}\} \min\{r_{\max}|V(G)|, |E(G)|\})$, together with at most $\mathcal{O}(r_{\max}|V(G)|)$ calls of the procedure **FSVecCon-biconnect** (at most $\mathcal{O}(r_{\max})$ calls for each block of G). Thus, the claimed time complexity follows. \square

5. Polynomiality in block graphs

Using Theorem 3, we can show that the **FREEVECCON** problem is polynomial in the class of block graphs, thus generalizing in two ways the polynomial-time solvability of **VECCON** on trees due to Boros et al. [4]. To obtain this result, it suffices to argue that **FREEVECCON** is polynomial on complete graphs. Given a complete graph $K = (V, E)$ with vertex requirements $r : V \rightarrow \mathbb{Z}_+$ and a free set $F \subseteq V$, a simple exchange argument applied separately to F and $V - F$ implies that there exists an optimal solution consisting of k largest requirement vertices from F and ℓ largest requirement vertices from $V \setminus F$, for some $k \in \{0, \dots, |F|\}$ and some $\ell \in \{0, \dots, |V| - |F|\}$. An optimal solution can thus be found by sorting the vertices in F and $V \setminus F$ according to their requirements, and checking which of the $\mathcal{O}(|V(K)|^2)$ pairs (k, ℓ) as above minimizes the value of $k + \ell$ subject to the constraint that k largest requirement vertices from F and ℓ largest requirement vertices from $V - F$ form a vector connectivity set for (K, F, r) . Therefore, Theorem 3 yields the following.

Theorem 4. *VECCON and FREEVECCON problems are solvable in polynomial time on block graphs.*

A similar approach can be used to show that, more generally, **FREEVECCON** is polynomial also for the class of *block-cactus graphs*, that is, graphs every block of which is either a complete graph or a cycle (see, e.g., [18]). Indeed, it can be seen that **FREEVECCON** is polynomially solvable on cycles. (We omit the easy proof.) Block graphs and block-cactus graphs give further examples of graph classes with arbitrarily long induced paths for which **VECCON** is polynomially solvable (cf. the discussion in [4]).

For instances on arbitrary graphs with bounded requirements we can show the following result.

Theorem 5. *The VEC CON problem can be solved in polynomial time if all requirements are at most 2.*

Proof. We work on the block tree \mathcal{T} of G . For each leaf block L with cut vertex v such that for every vertex $u \in V(L) \setminus \{v\}$, we have $r(u) \leq 1$, we delete from the graph the vertex set $V(L) \setminus \{v\}$. We repeat this operation as long as possible. Let \mathcal{L} be the set of leaves of such a pruned block tree, which we denote \mathcal{T}' . Our solution S contains exactly one vertex v_L from each block $L \in \mathcal{L}$ such that v_L is not a cut vertex of the pruned graph, unless $|V(\mathcal{T}')| = 1$, in which case the pruned graph is biconnected, and the optimal solution is of size at most 2.

Assuming $|V(\mathcal{T}')| > 1$, for each $L \in \mathcal{L}$ let $\mathcal{S}(L)$ be the set of blocks B removed from \mathcal{T} such that every block on the path from L to B in \mathcal{T} has also been removed. Clearly every deleted vertex v of G in a block of $\mathcal{S}(L)$ has requirement 1 or 0 and can be reached by a path from the vertex selected in $S \cap L$. For each remaining block B we have that B lies in \mathcal{T} on a path between two leaf blocks of \mathcal{T}' say $L_1, L_2 \in \mathcal{L}$. Hence, each vertex in B is reached by two disjoint paths, one starting in $S \cap L_1$ and one in $S \cap L_2$.

To argue optimality, consider for each $L \in \mathcal{L}$ the set V_L of all vertices in $V(G) \setminus V(L)$ from a block in $\mathcal{S}(L)$. By Proposition 1, we have that $V_L \cap S' \neq \emptyset$ for each feasible solution S' . Since for every two distinct $L_1, L_2 \in \mathcal{L}$, we have $V_{L_1} \cap V_{L_2} = \emptyset$, we conclude that any solution S' must satisfy $|S'| \geq |\mathcal{L}|$. \square

6. Conclusion

We conclude with some questions for future research related to the **VECCON** problem.

1. Recall that if in **VECCON** each path is restricted to be of length exactly 1, we get the **VECTOR DOMINATION** problem. If we set $r(v) = d(v)$ in the **VECTOR DOMINATION** problem, we get the **VERTEX COVER** problem. This motivates the following question: What is the computational complexity of **VECCON** on input instances such that $r(v) = d(v)$ for every vertex $v \in V$?

2. The VECCON problem is polynomial for trees, block graphs, and split graphs, all subclasses of chordal graphs. What is the complexity of VECCON in chordal graphs?
3. Cographs and split graphs do not have induced paths on 4 and 5 vertices, respectively. What is the complexity of VECCON in the class of P_k -free graphs for $k \geq 5$? In particular, is there a $k \geq 5$ such that VECCON is NP-complete for P_k -free graphs?
4. The VECCON problem is APX-hard if $R(V(G)) = 4$ and polynomial if $R(V(G)) \leq 2$. What is the complexity of VECCON if $R(V(G)) = 3$?
5. While the VECCON problem can be approximated in polynomial time within a factor of $\ln n + 2$ on all n -vertex graphs [4], Theorem 2 shows that there is no PTAS for VECCON unless $P = NP$. Nevertheless, the exact (in-)approximability of VECCON remains open, including the question from [4] whether VECCON admits a polynomial-time constant-factor approximation algorithm on general graphs.

Finally, we remark that VECCON is fixed-parameter tractable with respect to solution size [12, 13].

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